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Cohomological aspects of Abelian gauge theory

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Abstract. We discuss some aspects of cohomological properties of a two-dimensional free Abelian gauge theory in the framework of the BRST formalism. We derive the conserved and nilpotent BRST and co-BRST charges and express the Hodge decomposition theorem in terms of these charges and a conserved bosonic charge corresponding to the Laplacian operator. It is because of the topological nature of free $U(1)$ gauge theory that the Laplacian operator goes to zero when equations of motion are exploited. We derive two sets of topological invariants which are related to each other by a certain kind of duality transformation and express the Lagrangian density of this theory as the sum of terms that are BRST and co-BRST invariants. Mathematically, this theory captures together some of the key features of Witten- and Schwarz-type topological field theories.

1. Introduction

One of the key theorems in the mathematical aspects of cohomology is the celebrated Hodge decomposition theorem defined on a compact manifold. This theorem states that any arbitrary p-form f_p on this manifold can be decomposed into a harmonic form ω_p ($\Delta\omega_p = 0, \mathrm{d}\omega_p = 0, \delta\omega_p = 0$), an exact form $\mathrm{d}g_{p-1}$ and a co-exact form δh_{p+1}:

$$f_p = \omega_p + \mathrm{d}g_{p-1} + \delta h_{p+1} \tag{1.1}$$

where $\delta(= \pm * \mathrm{d} *)$ is the Hodge dual of d (with $\mathrm{d}^2 = 0, \delta^2 = 0$) and Laplacian Δ is defined as $\Delta = (\mathrm{d} + \delta)^2 = \mathrm{d}\delta + \delta\mathrm{d}$ [1–4]. So far, the analogue of d has been found as the local, conserved and nilpotent ($Q_B^2 = 0$) Becchi–Rouet–Stora–Tyutin (BRST) charge Q_B, which generates a nilpotent BRST symmetry for a locally gauge-invariant Lagrangian density in any arbitrary dimension of spacetime. The physical state condition $Q_B|\mathrm{phys}\rangle = 0$ leads to the annihilation of physical states in the quantum Hilbert space by the first-class constraints of the original gauge theory. This requirement is essential for the consistent quantization of a theory endowed with the first-class constraints (see, e.g., [5–10])†. It will be an interesting idea to explore the possibility of finding the *local* conserved charges corresponding to δ and Δ so that a complete physical understanding of BRST cohomology and Hodge decomposition can emerge in the quantum Hilbert space of states.

The purpose of this paper is to provide some physical interpretations to the analogues of δ and Δ in the language of nilpotent (for δ), local, covariant and continuous symmetry properties of a free $U(1)$ gauge theory described by the BRST invariant Lagrangian densities

† Attempts have also been made to discuss the second-class constraints in the framework of the BRST formalism (see, e.g., [11, 12] and references therein).

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and show that this theory is a tractable field-theoretical model for the Hodge theory in two (1+1) dimensions of spacetime. Some very interesting and illuminating attempts [13–16] have been made towards this goal for the Abelian as well as non-Abelian gauge theories in any arbitrary dimension of spacetime. However, the symmetry transformations turn out to be non-local and non-covariant. In the relativistic covariant formulation, the symmetry transformations turn out to be even non-nilpotent and they become nilpotent only when some restrictions are imposed [17]. We shall demonstrate that for the two-dimensional (2D) BRST invariant free $U(1)$ gauge theory, a conserved and nilpotent co(dual)-BRST charge Q_D (i.e. the analogue of δ) can be defined which corresponds to a new local, covariant, continuous and nilpotent symmetry transformation under which the gauge-fixing term $\delta A = (\partial \cdot A)$† remains invariant. This should be compared and contrasted with the usual BRST transformation under which the 2-form $F = \mathrm{d}A$ remains invariant in the $U(1)$ gauge theory. Further, we show that the anticommutator of both these charges $W = \{Q_B, Q_D\}$ is the analogue of the Laplacian operator Δ and it turns out to be the Casimir operator for the extended BRST algebra. We implement the Hodge decomposition theorem with these charges and show that the requirement of the annihilation of physical (harmonic) states by Q_B and Q_D is sufficient to gauge away both the degrees of freedom of a single photon in two dimensions. The ensuing theory becomes topological in nature (as there are no propagating degrees of freedom left in the theory) [18]. In the framework of BRST cohomology and Hodge decomposition theorem, this fact is encoded in rendering the Casimir operator W to go to zero ($W \to 0$) when equations of motion are exploited and all the fields are assumed to fall off rapidly at $x \to \pm\infty$. In contrast, for the 2D interacting $U(1)$ gauge theory, it has been shown that W does not go to zero on-shell because of the presence of matter degrees of freedom in the theory [19]. For the topological 2D free $U(1)$ gauge theory, we derive two sets of topological invariants with respect to both the conserved and nilpotent charges Q_B and Q_D. These invariants turn out to be connected with each other by a certain specific type of duality transformation.

The outline of our paper is as follows. In section 2, we set up the notation and sketch briefly the essentials of BRST formalism for $U(1)$ gauge theory in any arbitrary spacetime dimension. Section 3 is devoted to the derivation of the nilpotent and conserved (anti)dual-BRST charge and the Laplacian operator in two dimensions of spacetime. This is followed, in section 4, by the discussion of an extended BRST algebra which is constituted by six conserved charges. We discuss Hodge decomposition theorem and obtain two sets of topological invariants in section 5. Finally, we make some concluding remarks in section 6.

2. Preliminary: BRST invariant Lagrangians

We begin with the BRST invariant Lagrangian density ($\mathcal{L}_b$) for the $U(1)$ gauge theory in the Feynman gauge (see, e.g., [5–9])

$$\mathcal{L}_b = -\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} - \tfrac{1}{2} (\partial \cdot A)^2 - \mathrm{i} \partial_\mu \bar{C}\, \partial^\mu C \tag{2.1}$$

where the first term is the classical Maxwell Lagrangian density and the second and third terms are the gauge-fixing and Faddeev–Popov ghost terms, respectively. Here the $U(1)$ gauge connection A_μ is defined through the 1-form $A = A_\mu\, \mathrm{d}x^\mu$ and the curvature term $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ ($\mu, \nu = 0, 1, 2, \ldots D - 1$) is obtained from the 2-form $F = \mathrm{d}A$ in any D-dimensional flat Minkowski spacetime. Furthermore, the gauge-fixing term $(\partial \cdot A) =$

† Here the 1-form $A = A_\mu\, \mathrm{d}x^\mu$ defines the vector potential A_μ of the $U(1)$ gauge theory. Furthermore, it can be easily seen that the gauge-fixing term $(\partial \cdot A) = \delta A$ is the Hodge dual of the 2-form $F = \mathrm{d}A$ in the Abelian $U(1)$ gauge theory in any arbitrary dimension of spacetime (see, e.g., [2]).

$\partial_\mu A^\mu \equiv \delta A$, is the Hodge dual of the 2-form $F = \mathrm{d}A$ and $\bar{C}(C)$ are the anti(ghost) fields. The following on-shell ($\Box C = 0$) nilpotent ($\delta_b^2 = 0$) symmetry transformations

$$\begin{aligned} &\delta_b A_\mu = \eta \partial_\mu C \qquad \delta_b C = 0 \qquad \delta_b F_{\mu\nu} = 0 \\ &\delta_b \bar{C} = -\mathrm{i}\eta(\partial \cdot A) \qquad \delta_b(\partial \cdot A) = \eta \Box C \end{aligned} \tag{2.2}$$

lead to the derivation of a conserved and on-shell nilpotent BRST charge Q_b

$$Q_b = \int \mathrm{d}^{D-1}x \left[\partial_0(\partial \cdot A)C - (\partial \cdot A)\partial_0 C\right] \tag{2.3}$$

where η is an anticommuting ($\eta C = -C\eta$, $\eta\bar{C} = -\bar{C}\eta$) spacetime-independent infinitesimal parameter. Introduction of an auxiliary field B in the Lagrangian density (2.1)

$$\mathcal{L}_B = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + B(\partial \cdot A) + \tfrac{1}{2}B^2 - \mathrm{i}\partial_\mu \bar{C}\,\partial^\mu C \tag{2.4}$$

enables the validity of the off-shell nilpotent ($\delta_B^2 = 0$) symmetry transformations

$$\begin{aligned} &\delta_B A_\mu = \eta \partial_\mu C \qquad \delta_B F_{\mu\nu} = 0 \qquad \delta_B C = 0 \\ &\delta_B \bar{C} = \mathrm{i}\eta B \qquad \delta_B B = 0 \qquad \delta_B(\partial \cdot A) = \eta \Box C \end{aligned} \tag{2.5}$$

which lead to the existence of an off-shell nilpotent and conserved BRST charge

$$Q_B = \int \mathrm{d}^{(D-1)}x \left[B\dot{C} - \dot{B}C\right]. \tag{2.6}$$

The invariance of the ghost action $I_{FP} = -\mathrm{i}\int \mathrm{d}^D x\, \partial_\mu \bar{C}\,\partial^\mu C$ under discrete symmetry transformations: $C \to \pm\mathrm{i}\bar{C}$, $\bar{C} \to \pm\mathrm{i}C$ implies the existence of a conserved and nilpotent anti-BRST charge (Q_{AB}) which can be derived from the expressions (2.3) and (2.6) by the substitution $C \to \pm\mathrm{i}\bar{C}$. The continuous global symmetry invariance of the total action under the transformations $C \to \mathrm{e}^{-\lambda}C$, $\bar{C} \to \mathrm{e}^{\lambda}\bar{C}$, $A_\mu \to A_\mu$, $B \to B$ (where λ is a global parameter), leads to the derivation of the conserved ghost charge (Q_g),

$$Q_g = -\mathrm{i}\int \mathrm{d}^{(D-1)}x \left[C\dot{\bar{C}} + \bar{C}\dot{C}\right]. \tag{2.7}$$

Together, these conserved charges obey the following algebra:

$$\begin{aligned} &Q_B^2 = \tfrac{1}{2}\{Q_B, Q_B\} = 0 \qquad Q_{AB}^2 = \tfrac{1}{2}\{Q_{AB}, Q_{AB}\} = 0 \\ &\{Q_B, Q_{AB}\} = Q_B Q_{AB} + Q_{AB} Q_B = 0 \\ &\mathrm{i}[Q_g, Q_B] = +Q_B \qquad \mathrm{i}[Q_g, Q_{AB}] = -Q_{AB} \end{aligned} \tag{2.8}$$

where the canonical (anti)commutators for the BRST invariant Lagrangians are exploited for the derivation of the above algebra. This algebra is valid for $U(1)$ gauge theory in any arbitrary dimension of spacetime. It will be noticed that the anticommutator $\{Q_B, Q_{AB}\} = 0$ implies that the combined transformations $\delta_B\delta_{AB} + \delta_{AB}\delta_B$ acting on any field produce no transformation at all. Thus, anti-BRST charge is not the analogue of the dual (adjoint) exterior derivative (δ) for the $U(1)$ gauge theory†.

† It has been demonstrated in [20] that the anticommutator of the cohomologically higher-order BRST and anti-BRST charges is not zero and it leads to the definition of a cohomologically higher-order Laplacian operator for the compact non-Abelian Lie algebras.

3. Dual-BRST symmetry in two dimensions

In addition to the symmetries: $C \to \pm i\bar{C}$, $\bar{C} \to \pm iC$, the ghost action $-i\int d^2x\, \partial_\mu \bar{C}\, \partial^\mu C$ in 2D respects another symmetry; namely†,

$$\partial_\mu \to \pm i\varepsilon_{\mu\nu}\partial^\nu \qquad \varepsilon_{\mu\nu}\varepsilon^{\mu\lambda} = -\delta_\nu^\lambda. \tag{3.1}$$

It turns out that the total 2D Lagrangian density (2.1)

$$\mathcal{L}_b = \tfrac{1}{2}E^2 - \tfrac{1}{2}(\partial \cdot A)^2 - i\partial_\mu \bar{C}\, \partial^\mu C \tag{3.2}$$

remains invariant under the combination of the above two transformations because the ghost term remains invariant on its own and the kinetic energy term and gauge-fixing term exchange with each other:

$$\tfrac{1}{2}E^2 = \tfrac{1}{2}(\partial_0 A_1 - \partial_1 A_0)^2 \quad \rightleftharpoons \quad -\tfrac{1}{2}(\partial \cdot A)^2 = \tfrac{1}{2}(\partial_0 A_0 - \partial_1 A_1)^2. \tag{3.3}$$

Thus, in addition to the gauge BRST symmetry (2.2), we have an on-shell ($\Box \bar{C} = 0$) nilpotent ($\delta_d^2 = 0$) dual-BRST symmetry δ_d for the Lagrangian density (3.2)

$$\begin{aligned} &\delta_d A_\mu = -\eta\varepsilon_{\mu\nu}\partial^\nu \bar{C} \qquad \delta_d C = -i\eta E \\ &\delta_d E = \eta\Box\bar{C} \qquad \delta_d \bar{C} = 0 \qquad \delta_d(\partial \cdot A) = 0 \end{aligned} \tag{3.4}$$

which can be derived from (2.2) by the substitutions: $C \to +i\bar{C}$, $\partial_\mu \to +i\varepsilon_{\mu\nu}\partial^\nu$‡. We christen this symmetry as dual BRST because, in contrast to δ_B transformations where the electric field E is invariant, in the case of δ_D, it is the gauge-fixing term $(\partial \cdot A)$ that remains invariant§. Thus, we shall call the duality transformations for the Lagrangian density (3.2) those where $C \to \pm i\bar{C}$, $\bar{C} \to \pm iC$, $\partial_\mu \to \pm i\varepsilon_{\mu\nu}\partial^\nu$. Introducing an auxiliary field $\mathcal{B}$, the analogue of the Lagrangian density (2.4) can be written as

$$\mathcal{L}_{\mathcal{B}} = \mathcal{B}E - \tfrac{1}{2}\mathcal{B}^2 + B(\partial \cdot A) + \tfrac{1}{2}B^2 - i\partial_\mu \bar{C}\, \partial^\mu C \tag{3.5}$$

which respects the following off-shell nilpotent ($\delta_D^2 = 0$) dual-BRST symmetry:

$$\begin{aligned} &\delta_D A_\mu = -\eta\varepsilon_{\mu\nu}\partial^\nu \bar{C} \qquad \delta_D \bar{C} = 0 \qquad \delta_D C = -i\eta\mathcal{B} \qquad \delta_D \mathcal{B} = 0 \\ &\delta_D E = \eta\Box\bar{C} \qquad \delta_D(\partial \cdot A) = 0 \qquad \delta_D B = 0. \end{aligned} \tag{3.6}$$

This off-shell nilpotent dual-BRST transformations can be obtained from the transformations (2.5) (with the inclusion of $\delta_B \mathcal{B} = 0$) by the substitution: $C \to +i\bar{C}$, $\partial_\mu \to +i\varepsilon_{\mu\nu}\partial^\nu$, $B \to -i\mathcal{B}$, $\mathcal{B} \to -iB$. It can be checked that the off-shell nilpotent BRST and dual-BRST transformations (2.5) and (3.6) are connected with each other by

$$\begin{aligned} &C \to i\bar{C} \qquad E \to i(\partial \cdot A) \qquad B \to -i\mathcal{B} \\ &\bar{C} \to iC \qquad (\partial \cdot A) \to iE \qquad \mathcal{B} \to -iB \end{aligned} \tag{3.7}$$

which is a manifestation of the fact that the Lagrangian density (3.5) goes to itself under the above substitutions. Thus, for the Lagrangian density (3.5), the duality transformations are $C \to \pm i\bar{C}$, $\bar{C} \to \pm iC$, $\partial_\mu \to \pm i\varepsilon_{\mu\nu}\partial^\nu$, $B \to \mp i\mathcal{B}$, $\mathcal{B} \to \mp iB$‖. These continuous

† We adopt here the notation in which the 2D flat Minkowski metric is $\eta_{\mu\nu} = \text{diag}(+1, -1)$ and $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = \partial_0\partial_0 - \partial_1\partial_1$, $\dot{f} = \partial_0 f$, $F_{01} = \partial_0 A_1 - \partial_1 A_0 = E = F^{10}$, $\varepsilon_{01} = \varepsilon^{10} = +1$, $(\partial \cdot A) = \partial_0 A_0 - \partial_1 A_1$.

‡ Here and in what follows, we shall take only the (+) sign in the transformations: $C \to \pm i\bar{C}$, $\bar{C} \to \pm iC$, $\partial_\mu \to \pm i\varepsilon_{\mu\nu}\partial^\nu$. However, analogous statements will be valid if we take a (−) sign.

§ As per our definition in the introduction, the gauge-fixing term $\delta A = (\partial \cdot A)$ with $\delta = \pm * d *$ is the dual of the 2-form $F = dA$ which is the electric field E here in two dimensions.

‖ Note that we have taken the upper sign of these transformations in equation (3.7). However, the above statements are valid for the lower sign as well.

symmetries $\delta_{(d,D)}$ lead to the derivation of the following conserved and nilpotent ($Q^2_{(d,D)} = 0$) dual-BRST charge due to Noether theorem:

$$Q_{(d,D)} = \int \mathrm{d}x \, [E\dot{\bar{C}} - \dot{E}\bar{C}] \equiv \int \mathrm{d}x \, [\mathcal{B}\dot{\bar{C}} - \dot{\mathcal{B}}\bar{C}] \tag{3.8}$$

which generates (3.4) and (3.6) (i.e. $\delta_r \phi = -\mathrm{i}\eta[\phi, Q_r]_\pm, r = d, D$ and $(+)-$ stands for a (anti)commutator corresponding to (fermionic) bosonic ϕ). Due to the discrete symmetry invariance of the ghost action under $C \to \mathrm{i}\bar{C}, \bar{C} \to \mathrm{i}C$, we have the existence of a conserved and nilpotent antidual-BRST charge $Q_{(Ad,AD)}$ which can be derived from (3.8) by these substitutions (i.e. $C \to \mathrm{i}\bar{C}$).

It is obvious that Q_B and Q_D are the fermionic symmetry generators ($Q_B^2 = 0$, $Q_D^2 = 0$) for the Lagrangian density (3.5). Thus, the anticommutator of the two ($\{Q_B, Q_D\}$) will also be a symmetry generator. The corresponding bosonic transformation $\delta_W = \{\delta_B, \delta_D\}$ with the infinitesimal bosonic transformation parameter κ $(= -\mathrm{i}\eta\eta')$

$$\begin{aligned} &\delta_W A_\mu = \kappa(\partial_\mu \mathcal{B} + \varepsilon_{\mu\nu}\partial^\nu B) \qquad \delta_W B = 0 \qquad \delta_W \mathcal{B} = 0 \\ &\delta_W C = 0 \qquad \delta_W \bar{C} = 0 \qquad \delta_W(\partial \cdot A) = \kappa \Box \mathcal{B} \qquad \delta_W E = -\kappa \Box B \end{aligned} \tag{3.9}$$

is the symmetry of the above Lagrangian density (3.5) because $\delta_W \mathcal{L}_\mathcal{B} = \kappa(\partial_\mu[B\partial^\mu \mathcal{B} - \mathcal{B}\partial^\mu B])$. Here η and η' are the infinitesimal fermionic transformation parameters corresponding to δ_B and δ_D, respectively. The generator of the above symmetry transformation (and the analogue of the Laplacian operator) is a conserved charge (W) given by

$$W = \int \mathrm{d}x \, [\mathcal{B}\dot{B} - B\dot{\mathcal{B}}]. \tag{3.10}$$

This conserved quantity can be calculated directly from the anticommutator $\{Q_B, Q_D\}$ if we exploit the canonical (anti)commutators: $\{C(x,t), \dot{\bar{C}}(y,t)\} = \delta(x-y)$, $\{\bar{C}(x,t), \dot{C}(y,t)\} = -\delta(x-y)$, $[A_0(x,t), B(y,t)] = \mathrm{i}\delta(x-y)$, $[A_1(x,t), \mathcal{B}(y,t)] = \mathrm{i}\delta(x-y)$ and the rest of the (anti)commutators are zero. Here $\delta(x-y)$ is the Dirac delta function.

4. Extended BRST algebra

The set of all the conserved charges are the (anti)BRST, (anti)dual BRST, ghost and the W operator. Together, these charges for the 2D free $U(1)$ gauge theory are

$$\begin{aligned} &Q_B = \int \mathrm{d}x \, [B\dot{C} - \dot{B}C] \qquad Q_{AB} = \mathrm{i}\int \mathrm{d}x \, [B\dot{\bar{C}} - \dot{B}\bar{C}] \\ &Q_D = \int \mathrm{d}x \, [\mathcal{B}\dot{\bar{C}} - \dot{\mathcal{B}}\bar{C}] \qquad Q_{AD} = \mathrm{i}\int \mathrm{d}x [\mathcal{B}\dot{C} - \dot{\mathcal{B}}C] \\ &W = \int \mathrm{d}x \, [\mathcal{B}\dot{B} - B\dot{\mathcal{B}}] \qquad Q_g = -\mathrm{i}\int \mathrm{d}x \, [C\dot{\bar{C}} + \bar{C}\dot{C}]. \end{aligned} \tag{4.1}$$

If we exploit the covariant canonical (anti)commutators, these conserved charges obey the following extended BRST algebra:

$$\begin{aligned} &[W, Q_k] = 0 \qquad k = B, D, AB, AD, g \\ &Q_B^2 = Q_{AB}^2 = Q_D^2 = Q_{AD}^2 = 0 \\ &\{Q_B, Q_D\} = \{Q_{AB}, Q_{AD}\} = W \\ &\mathrm{i}[Q_g, Q_B] = +Q_B \qquad \mathrm{i}[Q_g, Q_{AB}] = -Q_{AB} \\ &\mathrm{i}[Q_g, Q_D] = -Q_D \qquad \mathrm{i}[Q_g, Q_{AD}] = +Q_{AD} \end{aligned} \tag{4.2}$$

and all the rest of the (anti)commutators turn out to be zero. A few remarks are in order. First of all, we see that the operator W is the Casimir operator for the whole algebra and its ghost number is zero. The ghost number of Q_B and Q_{AD} is +1 and that of Q_D and Q_{AB} is -1. Now given a state $|\psi\rangle$ in the quantum Hilbert space with the ghost number n (i.e. $\mathrm{i}Q_g|\psi\rangle = n|\psi\rangle$), it is straightforward, due to the above commutation relations, to check that the following relations are satisfied:

$$\begin{aligned} \mathrm{i}Q_g Q_B|\psi\rangle &= (n+1)Q_B|\psi\rangle \\ \mathrm{i}Q_g Q_D|\psi\rangle &= (n-1)Q_D|\psi\rangle \\ \mathrm{i}Q_g W|\psi\rangle &= nW|\psi\rangle \end{aligned} \tag{4.3}$$

which demonstrate that, whereas W keeps the ghost number of a state intact and unaltered, the operator Q_B increases the ghost number by one and Q_D reduces this number by one. This property is similar to the operation of a Laplacian, an exterior derivative and a dual exterior derivative on an n-form defined on a compact manifold. Thus, we see that the degree of the differential form is analogous to the ghost number in the Hilbert space, the differential form itself is analogous to the quantum state in the Hilbert space, a compact manifold has an analogy with the quantum Hilbert space and d, δ and $\Delta = d\delta + \delta d$ are Q_B, Q_D and W, respectively. It is a notable point that d and δ can also be identified with Q_{AB} and Q_{AD} in the BRST formalism.

5. Hodge decomposition theorem and topological invariants

It is obvious from the algebra (4.2) and the consideration of the ghost number of states ($Q_B|\psi\rangle$, $Q_D|\psi\rangle$ and $W|\psi\rangle$) in (4.3) that one can now implement the Hodge decomposition theorem in the language of BRST and dual-BRST charges (see, e.g., [3, 7, 8])

$$|\psi\rangle_n = |\omega\rangle_n + Q_B|\theta\rangle_{n-1} + Q_D|\chi\rangle_{n+1} \tag{5.1}$$

by which, any state $|\psi\rangle_n$ in the quantum Hilbert space with ghost number n can be decomposed into a harmonic state $|\omega\rangle_n$, a BRST exact state $Q_B|\theta\rangle_{n-1}$ and a dual-BRST exact state $Q_D|\chi\rangle_{n+1}$. To refine the BRST cohomology, however, we have to choose a representative state as the physical state from the total states of the quantum Hilbert space. Let us pick out here the physical state as the harmonic state ($|\mathrm{phys}\rangle = |\omega\rangle$) from the Hodge decomposed state (5.1). The number of such harmonic states is finite for a given physical theory as it represents the number of solutions to the Laplace equation (see, e.g., [2]). By definition, such a state would satisfy the following conditions:

$$W|\mathrm{phys}\rangle = 0 \qquad Q_B|\mathrm{phys}\rangle = 0 \qquad Q_D|\mathrm{phys}\rangle = 0. \tag{5.2}$$

Due to the simple form of the equations of motion ($\Box A_\mu = 0, \Box C = 0, \Box\bar{C} = 0$) for the basic fields in the theory, it is very convenient to express them in the normal modes [21]

$$\begin{aligned} A_\mu(x,t) &= \int \frac{\mathrm{d}k}{(2\pi)^{1/2}(2k^0)^{1/2}}\left[a_\mu(k)\mathrm{e}^{-\mathrm{i}k\cdot x} + a_\mu^\dagger(k)\mathrm{e}^{\mathrm{i}k\cdot x}\right] \\ C(x,t) &= \int \frac{\mathrm{d}k}{(2\pi)^{1/2}(2k^0)^{1/2}}\left[c(k)\mathrm{e}^{-\mathrm{i}k\cdot x} + c^\dagger(k)\mathrm{e}^{\mathrm{i}k\cdot x}\right] \\ \bar{C}(x,t) &= \int \frac{\mathrm{d}k}{(2\pi)^{1/2}(2k^0)^{1/2}}\left[b(k)\mathrm{e}^{-\mathrm{i}k\cdot x} + b^\dagger(k)\mathrm{e}^{\mathrm{i}k\cdot x}\right] \end{aligned} \tag{5.3}$$

where k_μ are the 2D momenta with the components $(k_0, k_1 = k)$. The on-shell nilpotent symmetry transformations (2.2) and (3.4), that are generated by the charges Q_b and Q_d, can now be exploited to yield (see, e.g., [21, 22] for details)

$$\begin{aligned} &[Q_b, a_\mu^\dagger(k)] = -k_\mu c^\dagger(k) && [Q_d, a_\mu^\dagger(k)] = \varepsilon_{\mu\nu} k^\nu b^\dagger(k) \\ &[Q_b, a_\mu(k)] = k_\mu c(k) && [Q_d, a_\mu(k)] = -\varepsilon_{\mu\nu} k^\nu b(k) \\ &\{Q_b, c^\dagger(k)\} = 0 && \{Q_d, c^\dagger(k)\} = \mathrm{i}\varepsilon^{\mu\nu} k_\mu a_\nu^\dagger \\ &\{Q_b, c(k)\} = 0 && \{Q_d, c(k)\} = -\mathrm{i}\varepsilon^{\mu\nu} k_\mu a_\nu \\ &\{Q_b, b^\dagger(k)\} = -\mathrm{i}k^\mu a_\mu^\dagger && \{Q_d, b^\dagger(k)\} = 0 \\ &\{Q_b, b(k)\} = +\mathrm{i}k^\mu a_\mu && \{Q_d, b(k)\} = 0. \end{aligned} \tag{5.4}$$

Similarly, the Casimir operator W generates the following commutation relations:

$$\begin{aligned} &[W, a_\mu^\dagger(k)] = \mathrm{i}k^2 \varepsilon_{\mu\nu} (a^\nu)^\dagger \qquad [W, a_\mu(k)] = -\mathrm{i}k^2 \varepsilon_{\mu\nu} a^\nu \\ &[W, c(k)] = [W, c^\dagger(k)] = [W, b(k)] = [W, b^\dagger(k)] = 0. \end{aligned} \tag{5.5}$$

We are now in a position to define the physical vacuum $|\mathrm{vac}\rangle$ as

$$\begin{aligned} &Q_b|\mathrm{vac}\rangle = Q_d|\mathrm{vac}\rangle = W|\mathrm{vac}\rangle = 0 \\ &a_\mu|\mathrm{vac}\rangle = c(k)|\mathrm{vac}\rangle = b(k)|\mathrm{vac}\rangle = 0. \end{aligned} \tag{5.6}$$

A single photon state $|e(k), \mathrm{vac}\rangle$ with polarization vector e_μ can be created from the physical vacuum by the application of a creation operator $\mathrm{e}^\mu a_\mu^\dagger|\mathrm{vac}\rangle \equiv |e(k), \mathrm{vac}\rangle$. The physicality criteria $Q_b|e(k), \mathrm{vac}\rangle = -(k \cdot e)c^\dagger(k)|\mathrm{vac}\rangle = 0$, $Q_d|e(k), \mathrm{vac}\rangle = \varepsilon_{\mu\nu} e^\mu k^\nu b^\dagger(k)|\mathrm{vac}\rangle = 0$ lead to the transversality $(k \cdot e = 0)$ of the photon and the condition $\varepsilon_{\mu\nu} e^\mu k^\nu = 0$ between e_μ and k_μ. Together, these conditions (due to the presence of extended symmetries) remove both the physical degrees of freedom of the 2D photon and imply the masslessness condition $k^2 = 0$ (see, e.g., [22] for more discussions).

The operation of the W operator on a single photon state (i.e. $W|e(k), \mathrm{vac}\rangle = -\mathrm{i}k^2 \varepsilon_{\mu\nu} e^\mu (a^\nu)^\dagger|\mathrm{vac}\rangle = 0$) implies the on-shell condition $(\Box A_\mu = 0 \to k^2 = 0)$ as well as the masslessness condition $(k^2 = 0)$ for the photon. The other relations, $k \cdot e = 0$, $\varepsilon_{\mu\nu} e^\mu k^\nu = 0$, emerging from the operation of Q_b and Q_d on a single photon state, are *unique* solutions to $k^2 = 0$. Thus, in a subtle way, $W|\mathrm{phys}\rangle = 0$ does imply the validity of $Q_b|\mathrm{phys}\rangle = 0$ and $Q_d|\mathrm{phys}\rangle = 0$. If basic symmetries are the central guiding principle, the operation of the W operator on a single physical photon state in 2D is superfluous (in some sense) because the symmetry corresponding to W can be derived from the symmetries generated by $Q_{(b,B)}$ and $Q_{(d,D)}$. This fact is encoded in the expression for the operator W (cf equation (4.1)) which can be re-expressed as

$$W = \int \mathrm{d}x \, \frac{\mathrm{d}}{\mathrm{d}x} \left[\tfrac{1}{2}\mathcal{B}^2 - \tfrac{1}{2}B^2\right] \to 0 \qquad \text{as} \quad x \to \pm\infty \tag{5.7}$$

due to the equation of motion $\partial_\mu \mathcal{B} + \varepsilon_{\mu\nu} \partial^\nu B = 0$. One cannot think of the off-shell validity of the expression for W in (4.1) because of the considerations of BRST cohomology. The presence of the two nilpotent symmetries corresponding to $Q_{(b,B)}$ and $Q_{(d,D)}$ and the requirement that: $Q_{(b,B)}|\mathrm{phys}\rangle = 0$, $Q_{(d,D)}|\mathrm{phys}\rangle = 0$, forces the physical 2D photon to always satisfy the on-shell $(\Box A_\mu = 0)$ as well as the mass-shell $(k^2 = 0)$ condition. Thus, there is no escape from the condition $W \to 0$ for a topological field theory where all the physical degrees of freedom are gauged away by symmetries alone. The topological nature of the theory is reflected by the presence of the topological invariants on the 2D manifold. The two sets of these invariants, with respect to both the conserved ($\dot{Q}_B = 0, \dot{Q}_D = 0$) and off-shell nilpotent ($Q_B^2 = 0, Q_D^2 = 0$) charges Q_B and Q_D, are

$$I_k[C_k] = \oint_{C_k} V_k \qquad J_k[C_k] = \oint_{C_k} W_k \qquad (k = 0, 1, 2) \tag{5.8}$$

where C_k are the k-dimensional homology cycles in the 2D manifold and k-form V_k and W_k for the 2D free $U(1)$ gauge theory are juxtaposed as

$$\begin{aligned} V_0 &= BC & W_0 &= \mathcal{B}\bar{C} \\ V_1 &= \big[BA_\mu + \mathrm{i}C\,\partial_\mu \bar{C}\big]\,\mathrm{d}x^\mu & W_1 &= \big[\bar{C}\varepsilon_{\mu\rho}\partial^\rho C - \mathrm{i}\mathcal{B}A_\mu\big]\,\mathrm{d}x^\mu \\ V_2 &= \mathrm{i}\big[A_\mu \partial_\nu \bar{C} - \tfrac{1}{2}\bar{C}F_{\mu\nu}\big]\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu & W_2 &= \mathrm{i}\big[\varepsilon_{\mu\rho}\partial^\rho C A_\nu + \tfrac{1}{2}C\varepsilon_{\mu\nu}(\partial\cdot A)\big]\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu. \end{aligned} \tag{5.9}$$

It can be seen that V_0 and W_0 are BRST ($\delta_B V_0 = 0$) and co-BRST invariant ($\delta_D W_0 = 0$) and V_2 and W_2 are closed ($dV_2 = 0$) and co-closed ($\delta W_2 = 0$), respectively. Using the canonical (anti)commutation relations with $\mathrm{i}Q_g$, it can be checked that the ghost numbers for (V_0, V_1, V_2) are $(+1, 0, -1)$ and that of (W_0, W_1, W_2) are $(-1, 0, +1)$, respectively. This fact can be succinctly expressed (for $k = 0, 1, 2$) as

$$\begin{aligned} \mathrm{i}[Q_g, V_k] &= (-1)^{1-k}(k-1)V_k \\ \mathrm{i}[Q_g, W_k] &= (-1)^{1-k}(1-k)W_k. \end{aligned} \tag{5.10}$$

These invariants (for $k = 1, 2$) obey the following relations (see, e.g., [18, 23, 24]):

$$\begin{aligned} \delta_B V_k &= \eta d V_{k-1} \qquad & d &= \mathrm{d}x^\mu\,\partial_\mu \\ \delta_D W_k &= \eta \delta W_{k-1} \qquad & \delta &= \mathrm{i}\,\mathrm{d}x^\mu\,\varepsilon_{\mu\nu}\partial^\nu \end{aligned} \tag{5.11}$$

where d and δ are the exterior and dual-exterior derivatives on the 2D compact manifold. Both these sets of topological invariants are related to each other by the duality transformations (3.7) as $I_k \to J_k$ under the substitutions: $B \to -\mathrm{i}\mathcal{B}$, $C \to \mathrm{i}\bar{C}$, $\partial_\mu \to \mathrm{i}\varepsilon_{\mu\nu}\partial^\nu$.

Using the on-shell nilpotent BRST and dual-BRST transformations (2.2) and (3.4), it will be interesting to verify that, modulo some total derivatives, the Lagrangian density (3.2) can be written as the sum of BRST and co-BRST invariant parts:

$$\eta\mathcal{L}_b = \tfrac{1}{2}\delta_d\big[\mathrm{i}EC\big] - \tfrac{1}{2}\delta_b\big[\mathrm{i}(\partial\cdot A)\bar{C}\big]. \tag{5.12}$$

The invariance of this Lagrangian density under BRST and dual-BRST transformations is easy to see because $\delta_b^2 = 0$, $\delta_d^2 = 0$ and $\{\delta_d, \delta_b\} \to 0$ as the Laplacian operator goes to zero ($W \to 0$) for the validity of the equations of motion. Furthermore, the expressions in the square brackets in (5.12) are BRST invariant (i.e. $\delta_b[\mathrm{i}EC] = 0$) and co-BRST invariant (i.e. $\delta_d[\mathrm{i}(\partial\cdot A)\bar{C}] = 0$). Using the fact that $Q_r (r = b, d)$ is the generator of transformation $\delta_r\phi = -\mathrm{i}\eta[\phi, Q_r]_\pm$, where $(+)-$ stands for the (anti)commutator corresponding to ϕ being (fermionic) bosonic in nature, it can be seen that (5.12) can be written as $\mathcal{L}_b = \{Q_d, S_1\} + \{Q_b, S_2\}$ for $S_1 = \frac{1}{2}EC$, $S_2 = -\frac{1}{2}(\partial\cdot A)\bar{C}$. This shows that the free $U(1)$ topological gauge field theory is similar *in form* to the Witten-type theories [24] but completely different in outlook from the Schwarz-type theories [25]. To be very precise, the free $U(1)$ topological gauge field theory is somewhat different from [24] too. This is mainly because of the fact that, in our discussions, there are two conserved and nilpotent charges with respect to which the topological invariants are defined, whereas in [24] there exists only a single BRST charge which is obtained due to the presence of topological shift and local gauge symmetries. In our discussions, there is no shift symmetry at all. Thus, from *symmetry point of view*, the 2D free $U(1)$ gauge theory is more like Schwarz-type theories. It can be seen, however, that the *symmetric* energy–momentum tensor ($T_{\mu\nu}$) for the Lagrangian density (3.2) (or (5.12))

$$T_{\mu\nu} = -\tfrac{1}{2}\big[\varepsilon_{\mu\rho}E + \eta_{\mu\rho}(\partial\cdot A)\big]\partial_\nu A^\rho - \tfrac{1}{2}\big[\varepsilon_{\nu\rho}E + \eta_{\nu\rho}(\partial\cdot A)\big]\partial_\mu A^\rho$$
$$-\mathrm{i}\partial_\mu\bar{C}\,\partial_\nu C - \mathrm{i}\partial_\nu\bar{C}\,\partial_\mu C - \eta_{\mu\nu}\mathcal{L}_b \tag{5.13}$$

has the *same form* as the Witten- and Schwarz-type topological field theories because it can be re-expressed as

$$T_{\mu\nu} = \{Q_b, V^{(1)}_{\mu\nu}\} + \{Q_d, V^{(2)}_{\mu\nu}\} \tag{5.14}$$

where the exact expression for V, in terms of the local fields, are

$$\begin{aligned} V^{(1)}_{\mu\nu} &= \tfrac{1}{2}\big[\partial_\mu\bar{C}A_\nu + \partial_\nu\bar{C}A_\mu + \eta_{\mu\nu}(\partial\cdot A)\bar{C}\big] \\ V^{(2)}_{\mu\nu} &= \tfrac{1}{2}\big[\partial_\mu C\varepsilon_{\nu\rho}A^\rho + \partial_\nu C\varepsilon_{\mu\rho}A^\rho - \eta_{\mu\nu}EC\big]. \end{aligned} \tag{5.15}$$

It can be checked that the partition functions as well as the expectation values of the BRST invariants, co-BRST invariants and the topological invariants are metric independent†. The key point to show this fact in the framework of BRST cohomology is the requirement that $Q_b|\mathrm{phys}\rangle = 0$ and $Q_d|\mathrm{phys}\rangle = 0$ (see, e.g., [18] for details) and the metric independence of the path-integral measure (see, e.g., [23]).

6. Conclusions

It is obvious that the usual nilpotent BRST transformations correspond to a symmetry in which the 2-form $F = \mathrm{d}A$ (e.g., electric field E in 2 D) of the $U(1)$ gauge theory remains invariant. The nilpotent dual-BRST charge is the generator of a transformation in which the gauge-fixing term $((\partial\cdot A) = \delta A)$ remains invariant. The anticommutator of these two transformations corresponds to a symmetry that is generated by the Casimir operator for the whole algebra. Under this conserved operator, it is the ghost term that remains invariant. Basically, the presence of BRST and dual-BRST symmetries imply the existence of two gauge symmetries: $e_\mu \to e_\mu + \alpha k_\mu$, $e_\mu \to e_\mu + \beta\varepsilon_{\mu\nu}k^\nu$ (for α and β arbitrary constants) in the theory. In this paper, these extended symmetries have been exploited together to gauge away the dynamical degrees of freedom of 2D photon so that this theory becomes topological. The form of the Lagrangian density (5.12), the appearance of symmetric energy–momentum tensor (5.14) and the existence of BRST and co-BRST invariants in (5.9) confirm the (topological) nature of the theory. In fact, it is a new type of topological field theory which captures together some of the salient features of both Witten- and Schwarz-type theories. It is an interesting venture to generalize these symmetries to 2D free (having no interaction with matter fields) [26] as well as interacting non-Abelian gauge theories. Furthermore, it will be nice to explore the physical impact of these kind of symmetries in the context of physical 4D interacting gauge theories. In fact, as a first preliminary step in this direction, it has been shown in [19] that the dual-BRST transformation $\delta_D A_\mu = -\eta\varepsilon_{\mu\nu}\partial^\nu\bar{C}$ on the Abelian gauge field corresponds to the chiral transformation on the Dirac fields for fermions in 2D interacting $U(1)$ gauge theory. Thus, the Adler–Bardeen–Jackiw (ABJ) anomalies appear in the theory for the proof of conservation laws at the quantum level. It is, therefore, expected that the full strength of BRST cohomology and the Hodge decomposition theorem might shed some light on the ABJ anomalies and provide a clue to the well known result that in 2D, the 'anomalous' gauge theory

† Here we have taken only the flat Minkowski metric. However, our arguments are valid even if we take into account a non-trivial metric. The metric independence of the measure has been shown in [23].

is consistent, unitary and amenable to particle interpretation [27, 28]. The insights gained in 2D might turn out to be useful for the generalization of Hodge decomposition to physical 4D gauge theories. These are some of the issues which are under investigation and the results will be reported elsewhere.

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